Fuzzy Set Theory : Soft Computing Course Lecture 29 – 34, notes, slides

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Fuzzy Set Theory

Soft Computing

Introduction to fuzzy set, topics : classical set theory, fuzzy set theory, crisp and non-crisp Sets representation, capturing uncertainty, examples. Fuzzy membership and graphic interpretation of fuzzy sets - small, prime numbers, universal, finite, infinite, empty space; Fuzzy Operations - inclusion, comparability, equality, complement, union, intersection, difference; Fuzzy properties related to union, intersection, distributivity, law of excluded middle, law of contradiction, and cartesian product. Fuzzy relations : definition, examples, forming fuzzy relations, projections of fuzzy relations, max-min and min-max compositions.

Fuzzy Set Theory

Soft Computing

Topics

(Lectures 29, 30, 31, 32, 33, 34 6 hours)

Slides

1. **Introduction to fuzzy Set** 03-10

   What is Fuzzy set? Classical set theory; Fuzzy set theory; Crisp and Non-crisp Sets : Representation; Capturing uncertainty, Examples

2. **Fuzzy set** 11-32

   Fuzzy Membership; Graphic interpretation of fuzzy sets : small, prime numbers, universal, finite, infinite, empty space;

   Fuzzy Operations : Inclusion, Comparability, Equality, Complement, Union, Intersection, Difference;

   Fuzzy Properties : Related to union – Identity, Idempotence, Associativity, Commutativity ; Related to Intersection – Absorption, Identity, Idempotence, Commutativity, Associativity; Additional properties - Distributivity, Law of excluded middle, Law of contradiction; Cartesian product .

3. **Fuzzy Relations** 33-41

   Definition of Fuzzy Relation, examples;

   Forming Fuzzy Relations – Membership matrix, Graphical form; Projections of Fuzzy Relations – first, second and global; Max-Min and Min-Max compositions.

4. **References** 42

Fuzzy Set Theory

What is Fuzzy Set ?

- The word "fuzzy" means "vagueness". Fuzziness occurs when the boundary of a piece of information is not clear-cut.

- Fuzzy sets have been introduced by Lotfi A. Zadeh (1965) as an extension of the classical notion of set.

- Classical set theory allows the membership of the elements in the set in binary terms, a bivalent condition - an element either belongs or does not belong to the set.

  Fuzzy set theory permits the gradual assessment of the membership of elements in a set, described with the aid of a membership function valued in the real unit interval [0, 1].

- **Example:**

  Words like young, tall, good, or high are fuzzy.

  - There is no single quantitative value which defines the term young.
  - For some people, age 25 is young, and for others, age 35 is young.
  - The concept young has no clean boundary.
  - Age 1 is definitely young and age 100 is definitely not young;
  - Age 35 has some possibility of being young and usually depends on the context in which it is being considered.

1. Introduction

In real world, there exists much fuzzy knowledge;
Knowledge that is vague, imprecise, uncertain, ambiguous, inexact, or probabilistic in nature.

Human thinking and reasoning frequently involve fuzzy information, originating from inherently inexact human concepts. Humans, can give satisfactory answers, which are probably true.

However, our systems are unable to answer many questions. The reason is, most systems are designed based upon classical set theory and two-valued logic which is unable to cope with unreliable and incomplete information and give expert opinions.

We want, our systems should also be able to cope with unreliable and incomplete information and give expert opinions. Fuzzy sets have been able provide solutions to many real world problems.

Fuzzy Set theory is an extension of classical set theory where elements have degrees of membership.

• Classical Set Theory

A Set is any well defined collection of objects. An object in a set is called an element or member of that set.

– Sets are defined by a simple statement describing whether a particular element having a certain property belongs to that particular set.

– Classical set theory enumerates all its elements using

$$A = \{ a_1 , a_2 , a_3 , a_4 , ... a_n \}$$

If the elements a_i $(i = 1, 2, 3, ... n)$ of a set **A** are subset of universal set **X**, then set **A** can be represented for all elements $x \in X$ by its **characteristic function**

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases}$$

– A set **A** is well described by a function called characteristic function.

This function, defined on the universal space **X**, assumes :

a value of **1** for those elements **x** that belong to set **A**, and

a value of **0** for those elements **x** that do not belong to set **A**.

The notations used to express these mathematically are

$$\left.\begin{array}{l} A : X \rightarrow [0, 1] \\ A(x) = 1 ,\ x \text{ is a member of } A \\ A(x) = 0 ,\ x \text{ is not a member of } A \end{array}\right\} \quad \textbf{Eq.(1)}$$

Alternatively, the set **A** can be represented for all elements $x \in X$ by its characteristic function $\mu_A(x)$ defined as

$$\mu_A(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise} \end{cases} \qquad \textbf{Eq.(2)}$$

– Thus in classical set theory $\mu_A(x)$ has only the values **0** ('false') and **1** ('true''). Such sets are called **crisp sets.**

- **Fuzzy Set Theory**

  Fuzzy set theory is an extension of classical set theory where elements have varying degrees of membership. A logic based on the two truth values, *True* and *False*, is sometimes inadequate when describing human reasoning. Fuzzy logic uses the whole interval between **0** (false) and **1** (true) to describe human reasoning.

  – A **Fuzzy Set** is any set that allows its members to have different degree of membership, called **membership function**, in the interval **[0 , 1]**.

  – The **degree of membership** or truth is not same as probability;
    - fuzzy truth is not likelihood of some event or condition.
    - fuzzy truth represents membership in vaguely defined sets;

  – **Fuzzy logic** is derived from fuzzy set theory dealing with reasoning that is approximate rather than precisely deduced from classical predicate logic.

  – Fuzzy logic is capable of handling inherently imprecise concepts.

  – Fuzzy logic allows in linguistic form the set membership values to imprecise concepts like **"slightly", "quite" and "very".**

  – Fuzzy set theory defines Fuzzy Operators on Fuzzy Sets.

- **Crisp and Non-Crisp Set**

  – As said before, in classical set theory, the **characteristic function** $\mu_A(x)$ of Eq.(2) has only values **0** ('false') and **1** ('true'').
  Such sets are **crisp sets.**

  – For Non-crisp sets the characteristic function $\mu_A(x)$ can be defined.
  - The characteristic function $\mu_A(x)$ of Eq. (2) for the crisp set is generalized for the Non-crisp sets.
  - This generalized characteristic function $\mu_A(x)$ of Eq.(2) is called **membership function**.

  Such Non-crisp sets are called **Fuzzy Sets.**

  – Crisp set theory is not capable of representing descriptions and classifications in many cases; In fact, Crisp set does not provide adequate representation for most cases.

  – The proposition of Fuzzy Sets are motivated by the need to capture and represent real world data with uncertainty due to imprecise measurement.

  – The uncertainties are also caused by vagueness in the language.

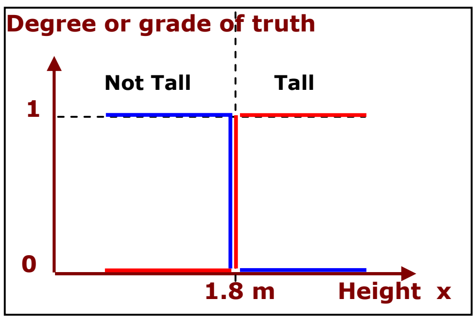
- **Representation of Crisp and Non-Crisp Set**

Example : Classify students for a basketball team

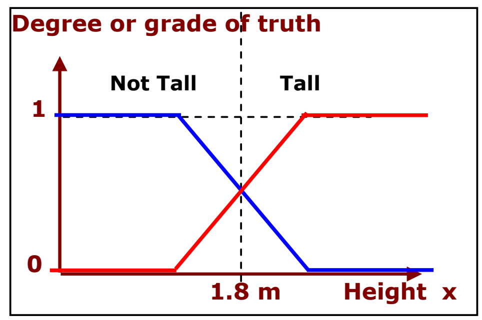
This example explains the grade of truth value.

- **tall students** qualify and **not tall students** do not qualify
- if students 1.8 m tall are to be qualified, then should we exclude a student who is $^1/_{10}$" less? or should we exclude a student who is 1" shorter?

■ Non-Crisp Representation to represent the notion of a tall person.



Crisp logic



Non-crisp logic

Fig. 1 Set Representation – Degree or grade of truth

A student of height 1.79m would belong to both tall and not tall sets with a particular degree of membership.

As the height increases the membership grade within the tall set would increase whilst the membership grade within the not-tall set would decrease.

- **Capturing Uncertainty**

Instead of avoiding or ignoring uncertainty, Lotfi Zadeh introduced Fuzzy Set theory that captures uncertainty.

- A fuzzy set is described by a **membership function** $\mu_A(x)$ of **A**.
  This membership function associates to each element $x_\sigma \in X$ a number as $\mu_A(x_\sigma)$ in the closed unit interval **[0, 1]**.

  The number $\mu_A(x_\sigma)$ represents the **degree of membership** of x_σ in **A**.
- The notation used for membership function $\mu_A(x)$ of a fuzzy set **A** is

  $$A : X \rightarrow [0, 1]$$
- Each membership function maps elements of a given universal base set **X** , which is itself a crisp set, into real numbers in **[0, 1]** .
- Example

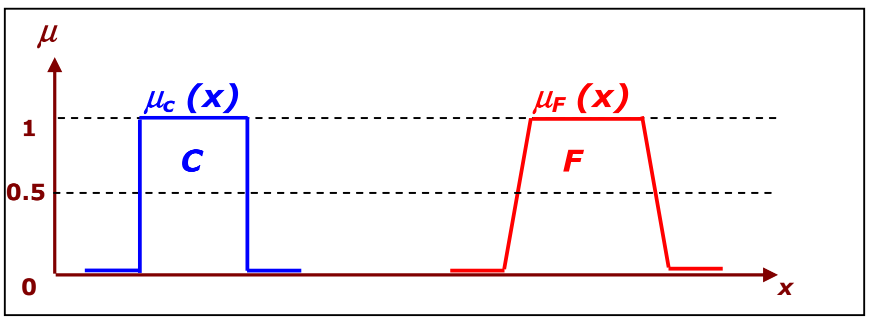


Fig. 2 Membership function of a Crisp set C and Fuzzy set F

- In the case of Crisp Sets the members of a set are :

  either out of the set, with membership of degree " **0** ",
  or in the set, with membership of degree " **1** ",

  Therefore, **Crisp Sets** $\subseteq$ **Fuzzy Sets**
  In other words, Crisp Sets are Special cases of Fuzzy Sets.

  [Continued in next slide]

- **Examples of Crisp and Non-Crisp Set**

Example 1: Set of prime numbers (a crisp set)

If we consider space **X** consisting of natural numbers ≤ **12**

ie **X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}**

Then, the set of prime numbers could be described as follows.

PRIME = {x contained in X | x is a prime number} = {2, 3, 5, 6, 7, 11}

Example 2: Set of SMALL (as non-crisp set)

A Set **X** that consists of SMALL cannot be described;

for example **1** is a member of SMALL and **12** is not a member of SMALL.

Set **A**, as SMALL, has un-sharp boundaries, can be characterized by a function that assigns a real number from the closed interval from **0** to **1** to each element **x** in the set **X**.

2. Fuzzy Set

A Fuzzy Set is any set that allows its members to have different degree of membership, called membership function, in the interval **[0 , 1]**.

- **Definition of Fuzzy set**

A **fuzzy set A**, defined in the universal space **X**, is a function defined in **X** which assumes values in the range **[0, 1]**.

A fuzzy set **A** is written as a set of pairs **{x, A(x)}** as

A = {{x , A(x)}} , x in the set X

where **x** is an element of the universal space **X**, and

A(x) is the value of the function **A** for this element.

The value **A(x)** is the **membership grade** of the element **x** in a fuzzy set **A**.

Example : Set **SMALL** in set **X** consisting of natural numbers **≤ to 12**.

Assume: **SMALL(1) = 1, SMALL(2) = 1, SMALL(3) = 0.9, SMALL(4) = 0.6,**
SMALL(5) = 0.4, SMALL(6) = 0.3, SMALL(7) = 0.2, SMALL(8) = 0.1,
SMALL(u) = 0 for u >= 9.

Then, following the notations described in the definition above :

Set SMALL = {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2},
{8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

Note that a fuzzy set can be defined precisely by associating with each **x** , its grade of membership in **SMALL**.

- **Definition of Universal Space**

Originally the universal space for fuzzy sets in fuzzy logic was defined only on the integers. Now, the universal space for fuzzy sets and fuzzy relations is defined with three numbers.

The first two numbers specify the start and end of the universal space, and the third argument specifies the increment between elements. This gives the user more flexibility in choosing the universal space.

Example : The fuzzy set of numbers, defined in the universal space
$X = \{ x_i \} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ is presented as
SetOption [FuzzySet, UniversalSpace → {1, 12, 1}]

2.1 Fuzzy Membership

A fuzzy set **A** defined in the universal space **X** is a function defined in **X** which assumes values in the range **[0, 1]**.

A fuzzy set **A** is written as a set of pairs **{x, A(x)}**.

A = {{x , A(x)}} , x in the set X

where **x** is an element of the universal space **X**, and

A(x) is the value of the function **A** for this element.

The value **A(x)** is the **degree of membership** of the element **x** in a fuzzy set **A**.

The Graphic Interpretation of fuzzy membership for the fuzzy sets : Small, Prime Numbers, Universal-space, Finite and Infinite UniversalSpace, and Empty are illustrated in the next few slides.

- **Graphic Interpretation of Fuzzy Sets SMALL**

The fuzzy set SMALL of small numbers, defined in the universal space

X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} is presented as

SetOption [FuzzySet, UniversalSpace → {1, 12, 1}]

The Set **SMALL** in set **X** is :

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

Therefore **SetSmall** is represented as

SetSmall = **FuzzySet [{{1,1},{2,1}, {3,0.9}, {4,0.6}, {5,0.4},{6,0.3}, {7,0.2}, {8, 0.1}, {9, 0}, {10, 0}, {11, 0}, {12, 0}}** , **UniversalSpace → {1, 12, 1}]**

FuzzyPlot [SMALL, AxesLable → {"X", "SMALL"}]

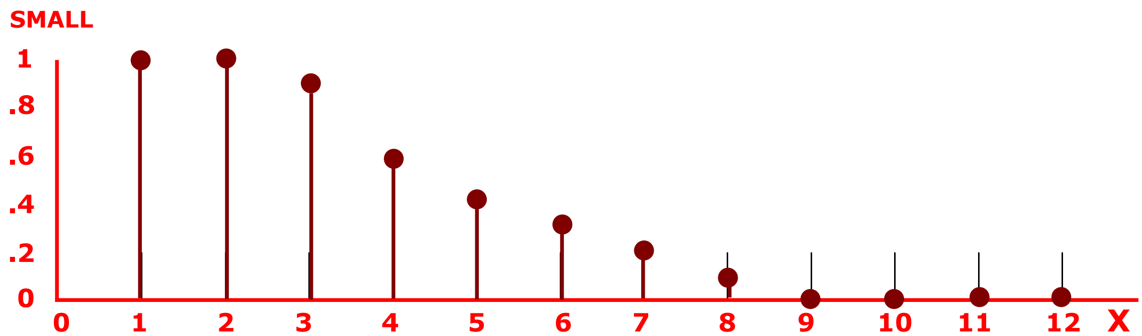


Fig Graphic Interpretation of Fuzzy Sets SMALL

- **Graphic Interpretation of Fuzzy Sets PRIME Numbers**

The fuzzy set PRIME numbers, defined in the universal space

X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} is presented as

SetOption [FuzzySet, UniversalSpace → {1, 12, 1}]

The Set **PRIME** in set **X** is :

PRIME = FuzzySet {{1, 0}, {2, 1}, {3, 1}, {4, 0}, {5, 1}, {6, 0}, {7, 1}, {8, 0}, {9, 0}, {10, 0}, {11, 1}, {12, 0}}

Therefore **SetPrime** is represented as

SetPrime = FuzzySet [{{1,0},{2,1}, {3,1}, {4,0}, {5,1},{6,0}, {7,1}, {8, 0}, {9, 0}, {10, 0}, {11, 1}, {12, 0}} , UniversalSpace → {1, 12, 1}]

FuzzyPlot [PRIME, AxesLable → {"X", "PRIME"}]

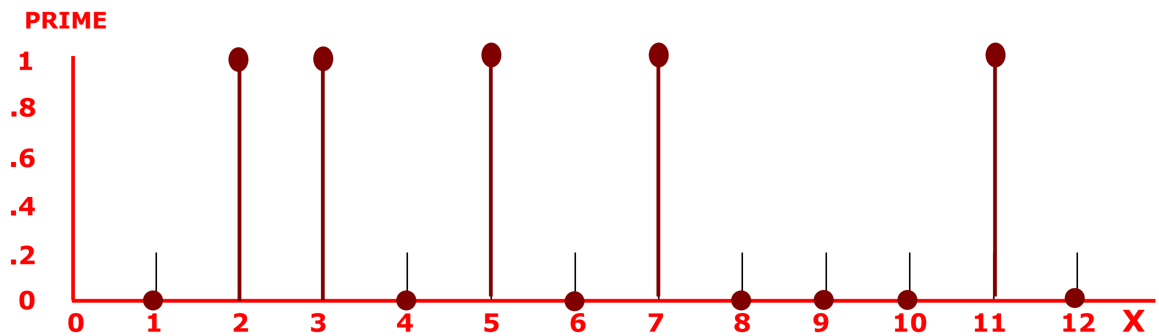


Fig Graphic Interpretation of Fuzzy Sets PRIME

- **Graphic Interpretation of Fuzzy Sets UNIVERSALSPACE**

In any application of sets or fuzzy sets theory, all sets are subsets of a fixed set called universal space or universe of discourse denoted by **X.** Universal space **X** as a fuzzy set is a function equal to **1** for all elements.

The fuzzy set **UNIVERSALSPACE** numbers, defined in the universal space **X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}** is presented as

SetOption [FuzzySet, UniversalSpace → {1, 12, 1}]

The Set **UNIVERSALSPACE** in set **X** is :

UNIVERSALSPACE = FuzzySet {{1, 1}, {2, 1}, {3, 1}, {4, 1}, {5, 1}, {6, 1}, {7, 1}, {8, 1}, {9, 1}, {10, 1}, {11, 1}, {12, 1}}

Therefore **SetUniversal** is represented as

SetUniversal = **FuzzySet [{{1,1},{2,1}, {3,1}, {4,1}, {5,1},{6,1}, {7,1}, {8, 1}, {9, 1}, {10, 1}, {11, 1}, {12, 1}} , UniversalSpace → {1, 12, 1}]**

FuzzyPlot [UNIVERSALSPACE, AxesLable → {"X", " UNIVERSAL SPACE "}]

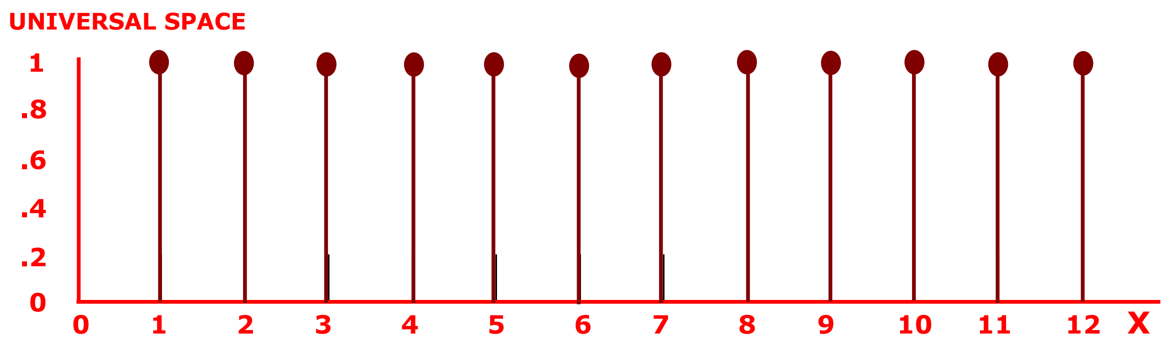


Fig Graphic Interpretation of Fuzzy Set UNIVERSALSPACE

- **Finite and Infinite Universal Space**

Universal sets can be finite or infinite.

Any universal set is finite if it consists of a specific number of different elements, that is, if in counting the different elements of the set, the counting can come to an end, else the set is infinite.

Examples:

1. Let **N** be the universal space of the days of the week.
   N = {Mo, Tu, We, Th, Fr, Sa, Su}. **N** is finite.
2. Let **M = {1, 3, 5, 7, 9, ...}.** **M** is infinite.
3. Let **L = {u | u is a lake in a city }.** **L** is finite.
   (Although it may be difficult to count the number of lakes in a city, but **L** is still a finite universal set.)

- **Graphic Interpretation of Fuzzy Sets EMPTY**

An empty set is a set that contains only elements with a grade of membership equal to **0**.

Example: Let EMPTY be a set of people, in Minnesota, older than 120.

The Empty set is also called the Null set.

The fuzzy set **EMPTY** , defined in the universal space

X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12} is presented as

SetOption [FuzzySet, UniversalSpace → {1, 12, 1}]

The Set **EMPTY** in set **X** is :

EMPTY = FuzzySet {{1, 0}, {2, 0}, {3, 0}, {4, 0}, {5, 0}, {6, 0}, {7, 0}, {8, 0}, {9, 0}, {10, 0}, {11, 0}, {12, 0}}

Therefore **SetEmpty** is represented as

SetEmpty = FuzzySet [{{1,0},{2,0}, {3,0}, {4,0}, {5,0},{6,0}, {7,0}, {8, 0}, {9, 0}, {10, 0}, {11, 0}, {12, 0}} , UniversalSpace → {1, 12, 1}]

FuzzyPlot [EMPTY, AxesLable → {"X", " UNIVERSAL SPACE "}]

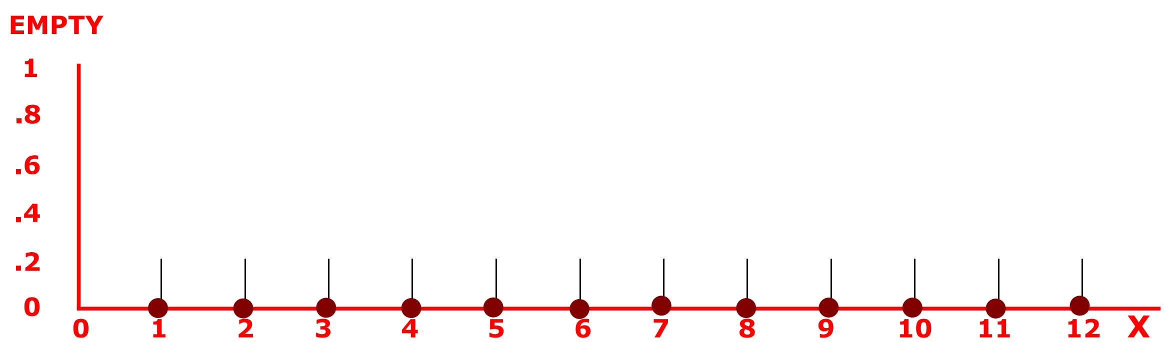


Fig Graphic Interpretation of Fuzzy Set EMPTY

2.2 Fuzzy Operations

A fuzzy set operations are the operations on fuzzy sets. The fuzzy set operations are generalization of crisp set operations. Zadeh [1965] formulated the fuzzy set theory in the terms of standard operations: Complement, Union, Intersection, and Difference.

In this section, the graphical interpretation of the following standard fuzzy set terms and the Fuzzy Logic operations are illustrated:

Inclusion : **FuzzyInclude [VERYSMALL, SMALL]**

Equality : **FuzzyEQUALITY [SMALL, STILLSMALL]**

Complement : **FuzzyNOTSMALL = FuzzyCompliment [Small]**

Union : **FuzzyUNION = [SMALL ∪ MEDIUM]**

Intersection : **FUZZYINTERSECTON = [SMALL ∩ MEDIUM]**

- **Inclusion**

Let **A** and **B** be fuzzy sets defined in the same universal space **X**.

The fuzzy set **A** is included in the fuzzy set **B** if and only if for every **x** in the set **X** we have **A(x)** ≤ **B(x)**

Example :

The fuzzy set **UNIVERSALSPACE** numbers, defined in the universal space **X = { x_i } = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}** is presented as

SetOption [FuzzySet, UniversalSpace → {1, 12, 1}]

The fuzzy set B SMALL

The Set **SMALL** in set **X** is :

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

Therefore **SetSmall** is represented as

SetSmall = **FuzzySet [{{1,1},{2,1}, {3,0.9}, {4,0.6}, {5,0.4},{6,0.3}, {7,0.2}, {8, 0.1}, {9, 0}, {10, 0}, {11, 0}, {12, 0}}** , **UniversalSpace → {1, 12, 1}]**

The fuzzy set A VERYSMALL

The Set **VERYSMALL** in set **X** is :

VERYSMALL = FuzzySet {{1, 1 }, {2, 0.8 }, {3, 0.7}, {4, 0.4}, {5, 0.2}, {6, 0.1}, {7, 0 }, {8, 0 }, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

Therefore **SetVerySmall** is represented as

SetVerySmall = **FuzzySet [{{1,1},{2,0.8}, {3,0.7}, {4,0.4}, {5,0.2},{6,0.1}, {7,0}, {8, 0}, {9, 0}, {10, 0}, {11, 0}, {12, 0}}** , **UniversalSpace → {1, 12, 1}]**

The Fuzzy Operation : Inclusion

Include [VERYSMALL, SMALL]

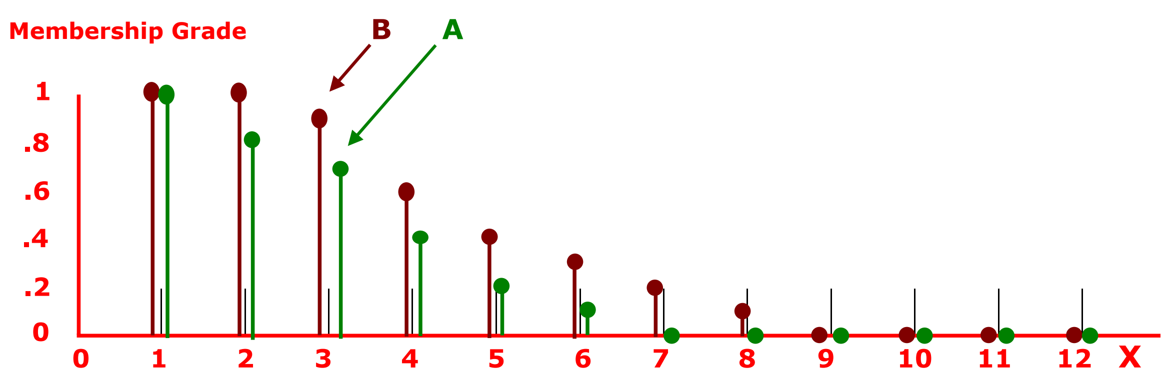


Fig Graphic Interpretation of Fuzzy Inclusion
FuzzyPlot [SMALL, VERYSMALL]

- **Comparability**

Two fuzzy sets **A** and **B** are comparable

if the condition **A** $\subset$ **B or B** $\subset$ **A** holds, ie,

if one of the fuzzy sets is a subset of the other set, they are comparable.

Two fuzzy sets **A** and **B** are incomparable

If the condition **A** $\not\subset$ **B or B** $\not\subset$ **A** holds.

Example 1:

Let **A = {{a, 1}, {b, 1}, {c, 0}}** **and**

B = {{a, 1}, {b, 1}, {c, 1}}.

Then **A** is comparable to **B**, since **A** is a subset of **B**.

Example 2 :

Let **C = {{a, 1}, {b, 1}, {c, 0.5}} and**

D = {{a, 1}, {b, 0.9}, {c, 0.6}}.

Then **C** and **D** are not comparable since

C is not a subset of **D** and

D is not a subset of **C**.

Property Related to Inclusion :

for all **x** in the set **X**, if **A(x)** $\subset$ **B(x)** $\subset$ **C(x), then accordingly A** $\subset$ **C.**

- **Equality**

Let **A** and **B** be fuzzy sets defined in the same space **X**.

Then **A** and **B** are equal, which is denoted **X = Y**

if and only if for all **x** in the set **X**, **A(x) = B(x).**

Example.

The fuzzy set B SMALL

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set A STILLSMALL

STILLSMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The Fuzzy Operation : Equality

Equality [SMALL, STILLSMALL]

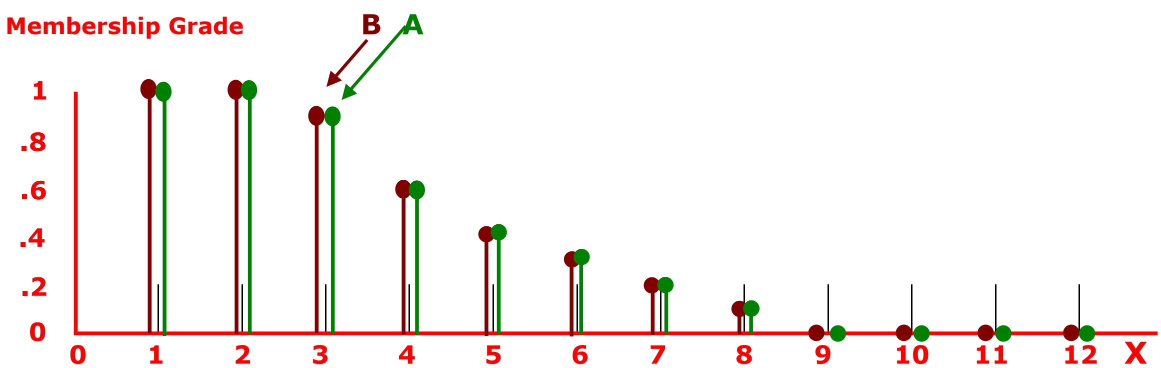


Fig Graphic Interpretation of Fuzzy Equality
FuzzyPlot [SMALL, STILLSMALL]

Note : If equality **A(x) = B(x)** is not satisfied even for one element **x** in the set **X**, then we say that **A** is not equal to **B**.

- **Complement**

Let **A** be a fuzzy set defined in the space **X**.

Then the fuzzy set **B** is a complement of the fuzzy set **A**, if and only if, for all **x** in the set **X**, **B(x) = 1 - A(x).**

The complement of the fuzzy set **A** is often denoted by **A'** or **Ac** or $\bar{A}$

Fuzzy Complement : **Ac(x) = 1 – A(x)**

Example 1.

The fuzzy set A SMALL

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set Ac NOTSMALL

NOTSMALL = FuzzySet {{1, 0 }, {2, 0 }, {3, 0.1}, {4, 0.4}, {5, 0.6}, {6, 0.7}, {7, 0.8}, {8, 0.9}, {9, 1 }, {10, 1 }, {11, 1}, {12, 1}}

The Fuzzy Operation : Compliment

NOTSMALL = Compliment [SMALL]

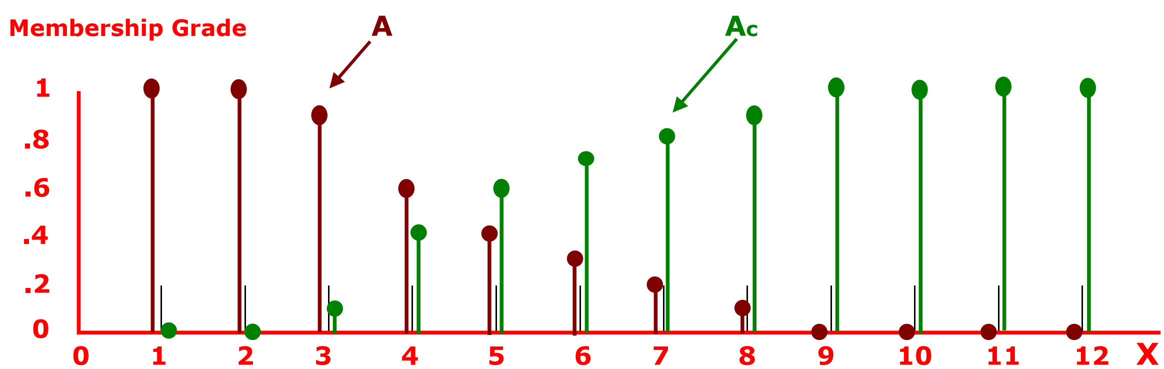


Fig Graphic Interpretation of Fuzzy Compliment
FuzzyPlot [SMALL, NOTSMALL]

Example 2.

The empty set Φ and the universal set **X**, as fuzzy sets, are complements of one another.

$$\Phi' = X \quad , \quad X' = \Phi$$

The fuzzy set B EMPTY

Empty = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0}, {5, 0}, {6, 0}, {7, 0}, {8, 0}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set A UNIVERSAL

Universal = FuzzySet {{1, 1 }, {2, 1 }, {3, 1}, {4, 1}, {5, 1}, {6, 1}, {7, 1}, {8, 1}, {9, 1 }, {10, 1 }, {11, 1}, {12, 1}}

The fuzzy operation : Compliment

EMPTY = Compliment [UNIVERSALSPACE]

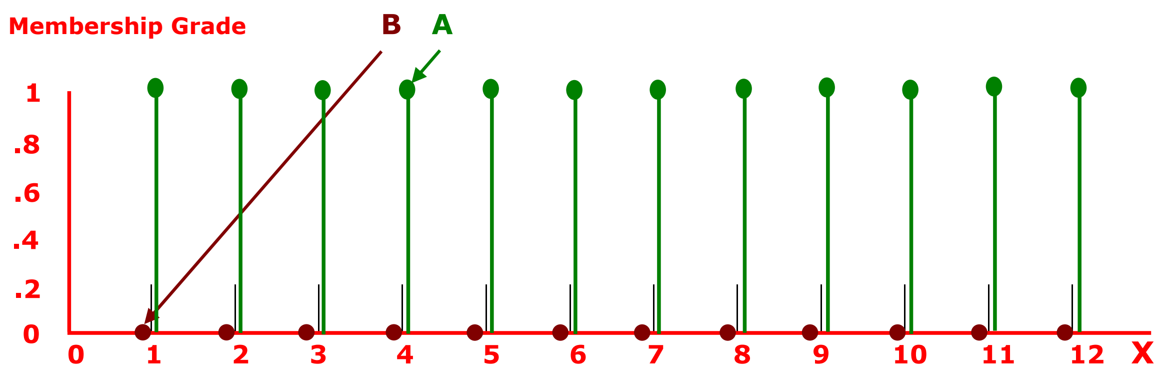


Fig Graphic Interpretation of Fuzzy Compliment
FuzzyPlot [EMPTY, UNIVERSALSPACE]

- **Union**

Let **A** and **B** be fuzzy sets defined in the space **X**.

The union is defined as the smallest fuzzy set that contains both **A** and **B**.

The union of **A** and **B** is denoted by **A ∪ B.**

The following relation must be satisfied for the union operation :

for all x in the set X, (A ∪ B)(x) = Max (A(x), B(x)).

Fuzzy Union : **(A ∪ B)(x) = max [A(x), B(x)] for all x ∈ X**

Example 1 : Union of Fuzzy **A** and **B**

A(x) = 0.6 and B(x) = 0.4 ∴ (A ∪ B)(x) = max [0.6, 0.4] = 0.6

Example 2 : Union of **SMALL** and **MEDIUM**

The fuzzy set A SMALL

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set B MEDIUM

MEDIUM = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0.2}, {5, 0.5}, {6, 0.8}, {7, 1}, {8, 1}, {9, 0.7 }, {10, 0.4 }, {11, 0.1}, {12, 0}}

The fuzzy operation : Union

FUZZYUNION = [SMALL ∪ MEDIUM]

SetSmallUNIONMedium = FuzzySet [{{1,1},{2,1}, {3,0.9}, {4,0.6}, {5,0.5}, {6,0.8}, {7,1}, {8, 1}, {9, 0.7}, {10, 0.4}, {11, 0.1}, {12, 0}} , UniversalSpace → {1, 12, 1}]

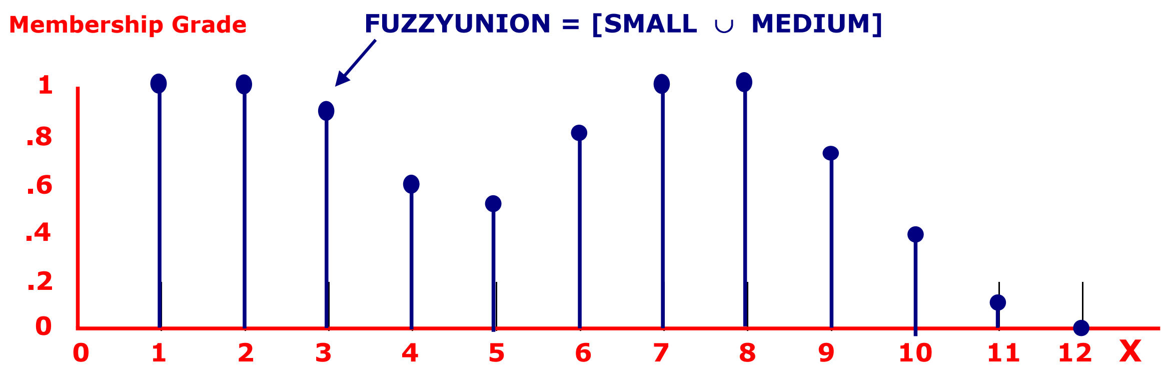


Fig Graphic Interpretation of Fuzzy Union
FuzzyPlot [UNION]

The notion of the union is closely related to that of the connective "or".

Let **A** is a class of "Young" men, **B** is a class of "Bald" men.

If "David is Young" or "David is Bald," then David is associated with the union of **A** and **B.** Implies David is a member of **A ∪ B**.

- **Intersection**

Let **A** and **B** be fuzzy sets defined in the space **X**. Intersection is defined as the greatest fuzzy set that include both **A** and **B**. Intersection of **A** and **B** is denoted by **A ∩ B.** The following relation must be satisfied for the intersection operation :

for all x in the set X, (A ∩ B)(x) = Min (A(x), B(x)).

Fuzzy Intersection : (A ∩ B)(x) = min [A(x), B(x)] for all x ∈ X

Example 1 : Intersection of Fuzzy **A** and **B**

A(x) = 0.6 and B(x) = 0.4 ∴ (A ∩ B)(x) = min [0.6, 0.4] = 0.4

Example 2 : Union of **SMALL** and **MEDIUM**

The fuzzy set A SMALL

SMALL = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0 }, {10, 0 }, {11, 0}, {12, 0}}

The fuzzy set B MEDIUM

MEDIUM = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0.2}, {5, 0.5}, {6, 0.8}, {7, 1}, {8, 1}, {9, 0.7 }, {10, 0.4 }, {11, 0.1}, {12, 0}}

The fuzzy operation : Intersection

FUZZYINTERSECTION = min [SMALL ∩ MEDIUM]

SetSmallINTERSECTIONMedium = FuzzySet [{{1,0},{2,0}, {3,0}, {4,0.2}, {5,0.4}, {6,0.3}, {7,0.2}, {8, 0.1}, {9, 0}, {10, 0}, {11, 0}, {12, 0}} , UniversalSpace → {1, 12, 1}]

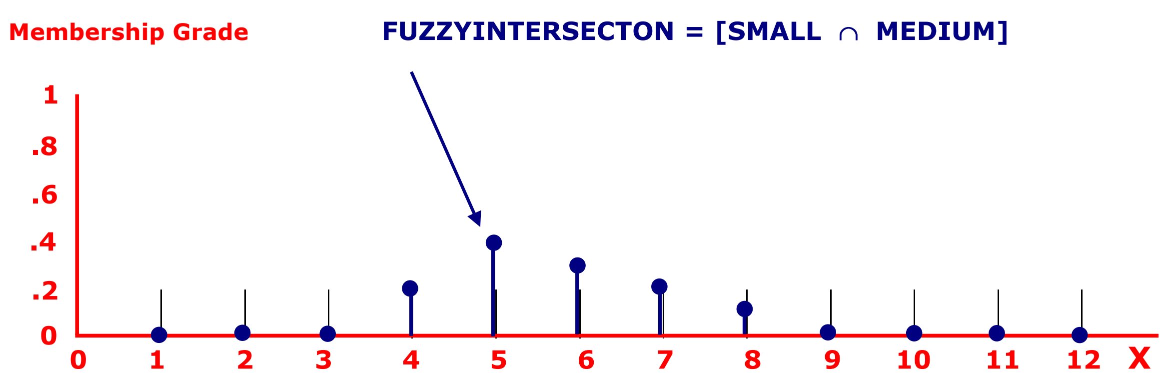


Fig Graphic Interpretation of Fuzzy Union
FuzzyPlot [INTERSECTION]

- **Difference**

Let **A** and **B** be fuzzy sets defined in the space **X**.

The difference of **A** and **B** is denoted by **A ∩ B'.**

Fuzzy Difference : **(A - B)(x) = min [A(x), 1- B(x)] for all x ∈ X**

Example : Difference of MEDIUM and SMALL

The fuzzy set A MEDIUM

MEDIUM = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0.2}, {5, 0.5}, {6, 0.8}, {7, 1}, {8, 1}, {9, 0.7 }, {10, 0.4 }, {11, 0.1}, {12, 0}}

The fuzzy set B SMALL

MEDIUM = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0.7 }, {10, 0.4 }, {11, 0}, {12, 0}}

Fuzzy Complement : Bc(x) = 1 – B(x)

The fuzzy set Bc NOTSMALL

NOTSMALL = FuzzySet {{1, 0 }, {2, 0 }, {3, 0.1}, {4, 0.4}, {5, 0.6}, {6, 0.7}, {7, 0.8}, {8, 0.9}, {9, 1 }, {10, 1 }, {11, 1}, {12, 1}}

The fuzzy operation : Difference by the definition of **Difference**

FUZZYDIFFERENCE = [MEDIUM ∩ SMALL']

SetMediumDIFFERECESmall = FuzzySet [{{1,0},{2,0}, {3,0}, {4,0.2}, {5,0.5}, {6,0.7}, {7,0.8}, {8, 0.9}, {9, 0.7}, {10, 0.4}, {11, 0.1}, {12, 0}} , UniversalSpace → {1, 12, 1}]

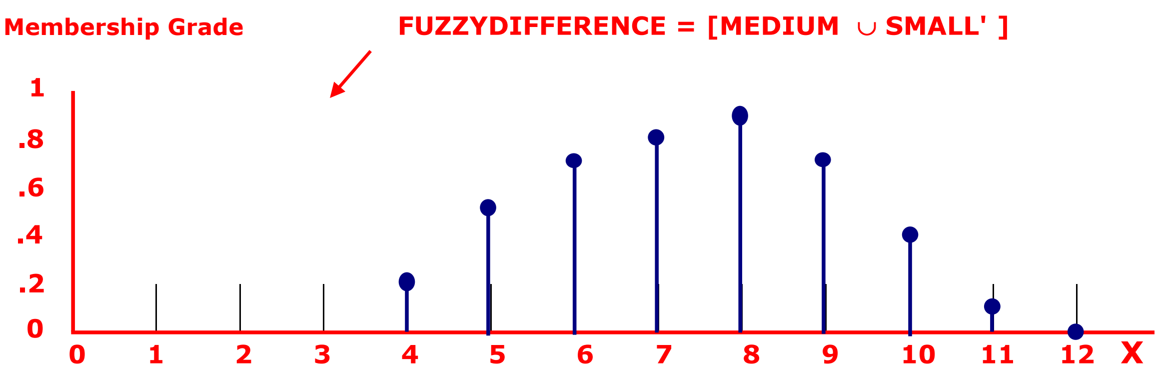


Fig Graphic Interpretation of Fuzzy Union
FuzzyPlot [UNION]

2.3 Fuzzy Properties

Properties related to Union, Intersection, Differences are illustrated below.

- **Properties Related to Union**

  The properties related to union are :

  Identity, Idempotence, Commutativity and Associativity.

  - **Identity:**

    $\mathbf{A \cup \Phi = A}$

    input = Equality [SMALL ∪ EMPTY , SMALL]

    output = True

    $\mathbf{A \cup X = X}$

    input = Equality [SMALL ∪ UnivrsalSpace , UnivrsalSpace]

    output = True

  - **Idempotence :**

    $\mathbf{A \cup A = A}$

    input = Equality [SMALL ∪ SMALL , SMALL]

    output = True

  - **Commutativity :**

    $\mathbf{A \cup B = B \cup A}$

    input = Equality [SMALL ∪ MEDIUM, MEDIUM ∪ SMALL]

    output = True

[Continued from previous slide]

- **Associativity:**

A ∪ (B∪ C) = (A∪ B) ∪ C

input = Equality [Small ∪ (Medium ∪ Big) , (Small ∪ Medium**)** ∪ Big]

output = True

Fuzzy Set Small , Medium , Big

Small = FuzzySet {{1, 1 }, {2, 1 }, {3, 0.9}, {4, 0.6}, {5, 0.4}, {6, 0.3}, {7, 0.2}, {8, 0.1}, {9, 0.7 }, {10, 0.4 }, {11, 0}, {12, 0}}

Medium = FuzzySet {{1, 0 }, {2, 0 }, {3, 0}, {4, 0.2}, {5, 0.5}, {6, 0.8}, {7, 1}, {8, 1}, {9, 0 }, {10, 0 }, {11, 0.1}, {12, 0}}

Big = FuzzySet [{{1,0}, {2,0}, {3,0}, {4,0}, {5,0}, {6,0.1}, {7,0.2}, {8,0.4}, {9,0.6}, {10,0.8}, {11,1}, {12,1}}]

Calculate Fuzzy relations :

(1) Medium ∪ Big = FuzzySet [{1,0},{2,0}, {3,0}, {4,0.2}, {5,0.5}, {6,0.8},{7,1}, {8, 1}, {9, 0.6}, {10, 0.8}, {11, 1}, {12, 1}]

(2) Small ∪ Medium = FuzzySet [{1,1},{2,1}, {3,0.9}, {4,0.6}, {5,0.5}, {6,0.8}, {7,1}, {8, 1}, {9, 0.7}, {10, 0.4}, {11, 0.1}, {12, 0}]

(3) Small ∪ (Medium ∪ Big) = FuzzySet [{1,1},{2,1}, {3,0.9}, {4,0.6}, {5,0.5}, {6,0.8}, {7,1}, {8, 1}, {9, 0.7}, {10, 0.8}, {11, 1}, {12, 1}]

(4) (Small ∪ Medium) ∪ Big] = FuzzySet [{1,1},{2,1}, {3,0.9}, {4,0.6}, {5,0.5}, {6,0.8}, {7,1}, {8, 1}, {9, 0.7},{10, 0.8}, {11, 1},{12, 1}]

Fuzzy set (3) and (4) proves Associativity relation

- **Properties Related to Intersection**

  Absorption, Identity, Idempotence, Commutativity, Associativity.

  - **Absorption by Empty Set :**

    $A \cap \Phi = \Phi$

    input = Equality [Small ∩ Empty , Empty]

    output = True

  - **Identity :**

    $A \cap X = A$

    input = Equality [Small ∩ UnivrsalSpace , Small]

    output = True

  - **Idempotence :**

    $A \cap A = A$

    input = Equality [Small ∩ Small , Small]

    output = True

  - **Commutativity :**

    $A \cap B = B \cap A$

    input = Equality [Small ∩ Big , Big ∩ Small]

    output = True

  - **Associativity :**

    $A \cap (B \cap C) = (A \cap B) \cap C$

    input = Equality [Small ∩ (Medium ∩ Big), (Small ∩ Medium) ∩ Big]

    output = True

- **Additional Properties**

Related to Intersection and Union

- **Distributivity:**

  $\mathbf{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$

  input = Equality [Small ∩ (Medium ∪ Big) ,
  (Small ∩ Medium) ∪ (Small ∩ Big)]

  output = True

- **Distributivity:**

  $\mathbf{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$

  input = Equality [Small ∪ (Medium ∩ Big) ,
  (Small ∪ Medium) ∩ (Small ∪ Big)]

  output = True

- **Law of excluded middle :**

  $\mathbf{A \cup A' = X}$

  input = Equality [Small ∪ NotSmall , UnivrsalSpace]

  output = True

- **Law of contradiction**

  $\mathbf{A \cap A' = \Phi}$

  input = Equality [Small ∩ NotSmall , EmptySpace]

  output = True

- **Cartesian Product Of Two Fuzzy Sets**

  - **Cartesian Product of two Crisp Sets**

    Let **A** and **B** be two crisp sets in the universe of discourse **X** and **Y**..

    The Cartesian product of **A** and **B** is denoted by **A x B**

    Defined as $A \times B = \{ (a, b) \mid a \in A,\ b \in B \}$

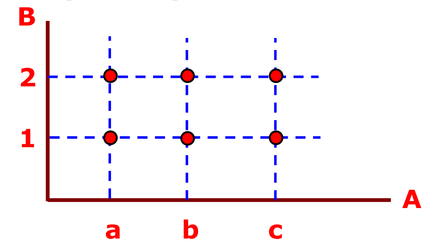
    Note : Generally $A \times B \neq B \times A$

    Example :

    Let $A = \{a, b, c\}$ and $B = \{1, 2\}$

    then $A \times B = \{ (a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2) \}$

    Graphic representation of A x B



  - **Cartesian product of two Fuzzy Sets**

    Let **A** and **B** be two fuzzy sets in the universe of discourse **X** and **Y**.

    The Cartesian product of **A** and **B** is denoted by **A x B**

    Defined by their membership function $\mu_A(x)$ and $\mu_B(y)$ as

    $$\mu_{A \times B}(x, y) = \min[\mu_A(x), \mu_B(y)] = \mu_A(x) \wedge \mu_B(y)$$

    or $$\mu_{A \times B}(x, y) = \mu_A(x)\, \mu_B(y)$$

    for all $x \in X$ and $y \in Y$

    Thus the Cartesian product **A x B** is a fuzzy set of ordered pair (x, y) for all $x \in X$ and $y \in Y$, with grade membership of (x, y) in $X \times Y$ given by the above equations .

    In a sense Cartesian product of two Fuzzy sets is a Fuzzy Relation.

3. Fuzzy Relations

Fuzzy Relations describe the degree of association of the elements;
Example : **"x is approximately equal to y".**

- Fuzzy relations offer the capability to capture the uncertainty and vagueness in relations between sets and elements of a set.
- Fuzzy Relations make the description of a concept possible.
- Fuzzy Relations were introduced to supersede classical crisp relations; It describes the total presence or absence of association of elements.

In this section, first the fuzzy relation is defined and then expressing fuzzy relations in terms of matrices and graphical visualizations. Later the properties of fuzzy relations and operations that can be performed with fuzzy relations are illustrated.

3.1 Definition of Fuzzy Relation

Fuzzy relation is a generalization of the definition of fuzzy set from 2-D space to 3-D space.

- **Fuzzy relation definition**

Consider a Cartesian product

$$A \times B = \{ (x , y) \mid x \in A, y \in B \}$$

where **A** and **B** are subsets of universal sets U_1 and U_2.

Fuzzy relation on **A x B** is denoted by **R** or **R(x , y)** is defined as the set

$$R = \{ ((x , y) , \mu_R (x , y)) \mid (x , y) \in A \times B , \mu_R (x , y) \in [0,1] \}$$

where $\mu_R(x , y)$ is a function in two variables called membership function.

- It gives the degree of membership of the ordered pair **(x , y)** in **R** associating with each pair **(x , y)** in **A x B** a real number in the interval **[0 , 1]**.
- The degree of membership indicates the degree to which **x** is in relation to **y**.

Note :

- Definition of fuzzy relation is a generalization of the definition of fuzzy set from the 2-D space $(x , , \mu_R (x))$ to 3-D space $((x , y) , \mu_R (x , y))$.
- Cartesian product **A x B** is a relation by itself between **x** and **y** .
- A fuzzy relation **R** is a sub set of R^3 namely

$$\{ ((x , y) , \mu_R (x , y)) \mid \in A \times B \times [0,1] \in U_1 \times U_2 \times [0,1] \}$$

- **Example of Fuzzy Relation**

$R = \{ ((x_1, y_1), 0)), \quad ((x_1, y_2), 0.1)), \quad ((x_1, y_3), 0.2)),$
$((x_2, y_1), 0.7)), ((x_2, y_2), 0.2)), \quad ((x_2, y_3), 0.3)),$
$((x_3, y_1), 1)), \quad ((x_3, y_2), 0.6)), \quad ((x_3, y_3), 0.2)),$

The relation can be written in matrix form as

$R \triangleq$

x \ y	y_1	Y_2	Y_3
x_1	0	0.1	0.2
X_2	0.7	0.2	0.3
X_3	1	0.6	0.2

where symbol $\triangleq$ means ' is defined as' and

the values in the matrix are the values of membership function:

$\mu_R(x_1, y_1) = 0 \qquad \mu_R(x_1, y_2) = 0.1 \qquad \mu_R(x_1, y_3) = 0.2$

$\mu_R(x_2, y_1) = 0.7 \qquad \mu_R(x_2, y_2) = 0.2 \qquad \mu_R(x_2, y_3) = 0.3$

$\mu_R(x_3, y_1) = 1 \qquad \mu_R(x_3, y_2) = 0.6 \qquad \mu_R(x_3, y_3) = 0.2$

Assuming $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ and $y_1 = 1$, $y_2 = 2$, $y_3 = 3$, the relation can be graphically represented by points in 3-D space (X, Y, μ) as :

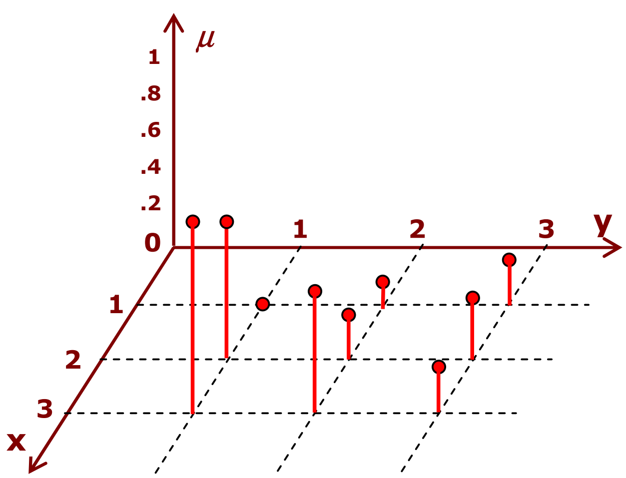


Fig Fuzzy Relation **R** describing **x** greater than **y**

Note : Since the values of the membership function **0.7, 1, 0.6** are in the direction of **x** below the major diagonal **(0, 0.2, 0.2)** in the matrix are grater than those **0.1, 0.2, 0.3** in the direction of **y**, we therefore say that the relation **R** describes **x** is grater than **y**.

3.2 Forming Fuzzy Relations

Assume that **V** and **W** are two collections of objects.

A fuzzy relation is characterized in the same way as it is in a fuzzy set.

- The first item is a list containing element and membership grade pairs,

  $\{\{v_1, w_1\}, R_{11}\}, \{\{v_1, w_2\}, R_{12}\}, \dots, \{\{v_n, w_m\}, R_{nm}\}\}$.

  where $\{v_1, w_1\}, \{v_1, w_2\}, \dots, \{v_n, w_m\}$ are the elements of the relation are defined as ordered pairs, and $\{R_{11}, R_{12}, \dots, R_{nm}\}$ are the membership grades of the elements of the relation that range from 0 to 1, inclusive.

- The second item is the universal space; for relations, the universal space consists of a pair of ordered pairs,

  $\{\{V_{min}, V_{max}, C_1\}, \{W_{min}, W_{max}, C_2\}\}$.

  where the first pair defines the universal space for the first set and the second pair defines the universal space for the second set.

Example showing how fuzzy relations are represented

Let V = {1, 2, 3} and W = {1, 2, 3, 4}.

A fuzzy relation **R** is, a function defined in the space **V x W**, which takes values from the interval [0, 1] , expressed as **R : V x W → [0, 1]**

R = FuzzyRelation [{{{1, 1}, 1}, {{1, 2}, 0.2}, {{1, 3}, 0.7}, {{1, 4}, 0}, {{2, 1}, 0.7}, {{2, 2}, 1}, {{2, 3}, 0.4}, {{2, 4}, 0.8}, {{3, 1}, 0}, {{3, 2}, 0.6}, {{3, 3}, 0.3}, {{3, 4}, 0.5}, UniversalSpace → {{1, 3, 1}, {1, 4, 1}}]

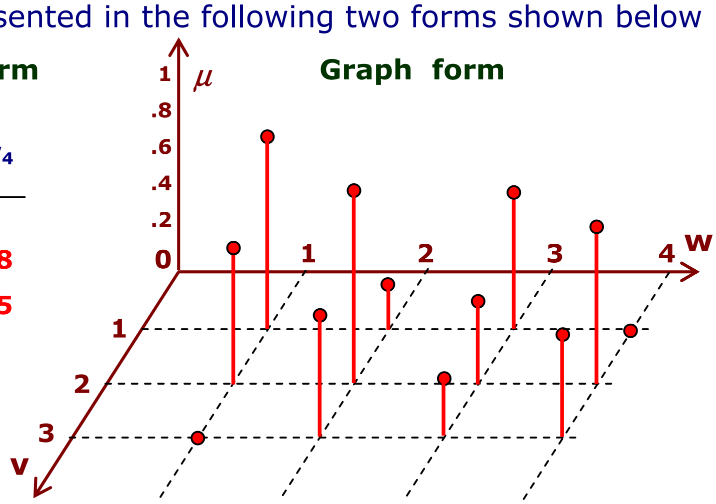
This relation can be represented in the following two forms shown below

Membership matrix form

$R \triangleq$

v \ w	w_1	w_2	w_3	w_4
v_1	1	0.2	0.7	0
v_2	0.7	1	0.4	0.8
v_3	0	0.6	0.3	0.5

Graph form



Vertical lines represent membership grades

Elements of fuzzy relation are ordered pairs $\{v_i, w_j\}$, where v_i is first and w_j is second element. The membership grades of the elements are represented by the heights of the vertical lines.

3.3 Projections of Fuzzy Relations

Definition : A fuzzy relation on **A x B** is denoted by **R** or **R(x , y)** is defined as the set

$$R = \{ ((x , y) , \mu_R (x , y)) \mid (x , y) \in A \times B , \mu_R (x , y) \in [0,1] \}$$

where $\mu_R (x , y)$ is a function in two variables called membership function. The first, the second and the total projections of fuzzy relations are stated below.

- **First Projection of R :** defined as

$$R^{(1)} = \{(x) , \mu_R^{(1)} (x , y))\}$$

$$= \{(x) , \max_Y \mu_R (x , y)) \mid (x , y) \in A \times B \}$$

- **Second Projection of R :** defined as

$$R^{(2)} = \{(y) , \mu_R^{(2)} (x , y))\}$$

$$= \{(y) , \max_X \mu_R (x , y)) \mid (x , y) \in A \times B \}$$

- **Total Projection of R :** defined as

$$R^{(T)} = \max_X \max_Y \{\mu_R (x , y) \mid (x , y) \in A \times B \}$$

Note : In all these three expression

$\max_Y$ means **max** with respect to **y** while **x** is considered fixed

$\max_X$ means **max** with respect to **x** while **y** is considered fixed

The Total Projection is also known as Global projection

- **Example : Fuzzy Projections**

The Fuzzy Relation **R** together with First, Second and Total Projection of **R** are shown below.

$R \triangleq$

x \ y	y_1	y_2	y_3	y_4	y_5	$R^{(1)}$		
x_1	0.1	0.3	1	0.5	0.3	1		
x_2	0.2	0.5	0.7	0.9	0.6	0.9		
x_3	0.3	0.6	1	0.8	0.2	1		
$R^{(2)}$	0.3	0.6	1	0.9	0.6	1	=	$R^{(T)}$

Note :

For $R^{(1)}$ select $\max\limits_{Y}$ means **max** with respect to **y** while **x** is considered fixed

For $R^{(2)}$ select $\max\limits_{x}$ means **max** with respect to **x** while **y** is considered fixed

For $R^{(T)}$ select **max** with respect to $R^{(1)}$ and $R^{(2)}$

The Fuzzy plot of these projections are shown below.

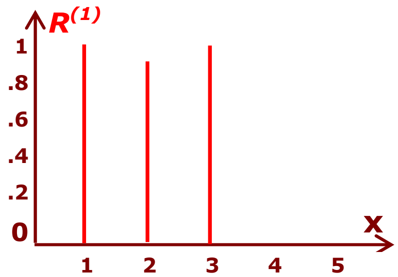


Fig Fuzzy plot of 1st projection $R^{(1)}$

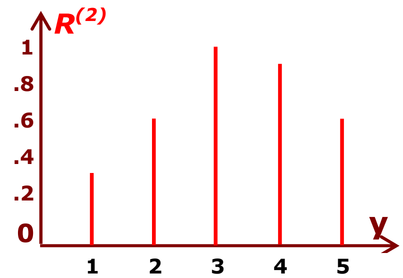


Fig Fuzzy plot of 2nd projection $R^{(2)}$

3.4 Max-Min and Min-Max Composition

The operation composition combines the fuzzy relations in different variables, say **(x , y)** and **(y , z)** ; $x \in A$, $y \in B$, $z \in C$.

Consider the relations :

$$R_1(x , y) = \{ ((x , y) , \mu_{R1}(x , y)) \mid (x , y) \in A \times B \}$$

$$R_2(y , z) = \{ ((y , y) , \mu_{R1}(y , z)) \mid (y , z) \in B \times C \}$$

The domain of R_1 is $A \times B$ and the domain of R_2 is $B \times C$

- **Max-Min Composition**

Definition : The Max-Min composition denoted by $R_1 \circ R_2$ with membership function $\mu_{R1 \circ R2}$ defined as

$$R_1 \circ R_2 = \{ ((x , z) , \max_{Y}(\min (\mu_{R1}(x , y) , \mu_{R2}(y , z))))\} ,$$

$$(x , z) \in A \times C , \; y \in B$$

Thus $R_1 \circ R_2$ is relation in the domain $A \times C$

An example of the composition is shown in the next slide.

- **Example : Max-Min Composition**

Consider the relations $R_1(x , y)$ and $R_2(y , z)$ as given below.

$R_1 \triangleq$

y / x	y_1	y_2	y_3
x_1	0.1	0.3	0
x_2	0.8	1	0.3

$R_2 \triangleq$

z / y	z_1	z_2	z_3
y_1	0.8	0.2	0
y_2	0.2	1	0.6
y_3	0.5	0	0.4

Note : Number of columns in the first table and second table are equal.

Compute max-min composition denoted by $R_1 \circ R_2$:

Step -1 Compute **min** operation (definition in previous slide).

Consider row x_1 and column z_1 , means the pair (x_1 , z_1) for all y_j , $j = 1, 2, 3$, and perform **min** operation

$\min (\mu_{R1} (x_1 , y_1) , \mu_{R2} (y_1 , z_1)) = \min (0.1, 0.8) = 0.1,$

$\min (\mu_{R1} (x_1 , y_2) , \mu_{R2} (y_2 , z_1)) = \min (0.3, 0.2) = 0.2,$

$\min (\mu_{R1} (x_1 , y_3) , \mu_{R2} (y_3 , z_1)) = \min (0, 0.5) = 0,$

Step -2 Compute **max** operation (definition in previous slide).

For $x = x_1$, $z = z_1$, $y = y_j$, $j = 1, 2, 3,$

Calculate the grade membership of the pair (x_1 , z_1) as

$\{ (x_1 , z_1) , \max ((\min (0.1, 0.8), \min (0.3, 0.2), \min (0, 0.5))$

i.e. $\{ (x_1 , z_1) , \max(0.1, 0.2, 0) \}$

i.e. $\{ (x_1 , z_1) , 0.2 \}$

Hence the grade membership of the pair (x_1 , z_1) is **0.2** .

Similarly, find all the grade membership of the pairs

$(x_1 , z_2) , (x_1 , z_3) , (x_2 , z_1) , (x_2 , z_2) , (x_2 , z_3)$

The final result is

$R_1 \circ R_2 =$

z / x	z_1	z_2	z_3
x_1	0.1	0.3	0
x_2	0.8	1	0.3

Note : If tables R_1 and R_2 are considered as matrices, the operation composition resembles the operation multiplication in matrix calculus linking row by columns. After each cell is occupied max-min value (the product is replaced by min, the sum is replaced by max).

- **Example : Min-Max Composition**

The min-max composition is similar to max-min composition with the difference that the roll of max and min are interchanged.

Definition : The max-min composition denoted by $R_1 \square R_2$ with membership function $\mu_{R1 \square R2}$ is defined by

$$R_1 \square R_2 = \{ ((x , z) , \min_{y} (\max (\mu_{R1} (x , y) , \mu_{R2} (y , z))))\} ,$$
$$(x , z) \in A \times C , \; y \in B$$

Thus $R_1 \square R_2$ is relation in the domain $A \times C$

Consider the relations $R_1(x , y)$ and $R_2(y , z)$ as given by the same relation of previous example of max-min composition, that is

$R_1 \triangleq$

x \ y	y_1	y_2	y_3
x_1	0.1	0.3	0
x_2	0.8	1	0.3

$R_2 \triangleq$

y \ z	z_1	z_2	z_3
y_1	0.8	0.2	0
y_2	0.2	1	0.6
y_3	0.5	0	0.4

After computation in similar way as done in the case of max-min composition, the final result is

$R_1 \square R_2 =$

x \ z	z_1	z_2	z_3
x_1	0.3	0	0.1
x_2	0.5	0.4	0.4

- **Relation between Max-Min and Min-Max Compositions**

The Max-Min and Min-Max Compositions are related by the formula

$$\overline{R_1} \circ \overline{R_2} = \overline{R_1 \square R_2}$$

4. References : Textbooks

1. *"Neural Network, Fuzzy Logic, and Genetic Algorithms - Synthesis and Applications", by S. Rajasekaran and G.A. Vijayalaksmi Pai, (2005), Prentice Hall, Chapter 6, page 157-186.*

2. *"Soft Computing and Intelligent Systems Design - Theory, Tools and Applications", by Fakhreddine karray and Clarence de Silva (2004), Addison Wesley, chapter 2, page 57-135.*

3. *"An Introduction to Neural Networks", by James A. Anderson, (1997), MIT Press, Chapter 1- 17, page 1-585.*

4. *"Fuzzy Sets and Fuzzy Logic: Theory and Applications", by George J. Klir and Bo Yuan, (1995), Prentice Hall, Chapter 1-9, page 1-278.*

5. *"Introduction To Fuzzy Sets And Fuzzy Logic", by M Ganesh, (2008), Prentice-hall, Chapter 1-8, page 1- 166.*

6. *"Fuzzy Logic: Intelligence, Control, and Information", by John Yen, Reza Langari, (1999), Prentice Hall, Chapter 1-7, page 1-167.*

7. *"Fuzzy Logic with Engineering Applications", by Timothy Ross, (2004), John Wiley & Sons Inc, Chapter 1-4 , page 1-114.*

8. *"Fuzzy Logic and Neuro Fuzzy Applications Explained", by Constantin Von Altrock, (1995), Prentice Hall, Chapter 1-2, page 1-28.*

9. *Related documents from open source, mainly internet. An exhaustive list is being prepared for inclusion at a later date.*